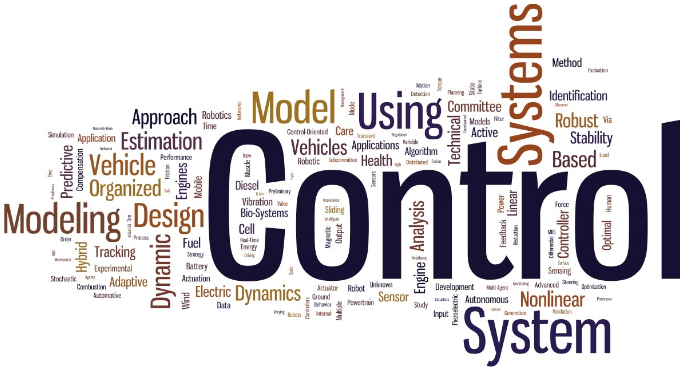
POLYTECH NANCY

UNIVERSITÉ DE LORRAINE



Automatique

PID controller design

A few examples

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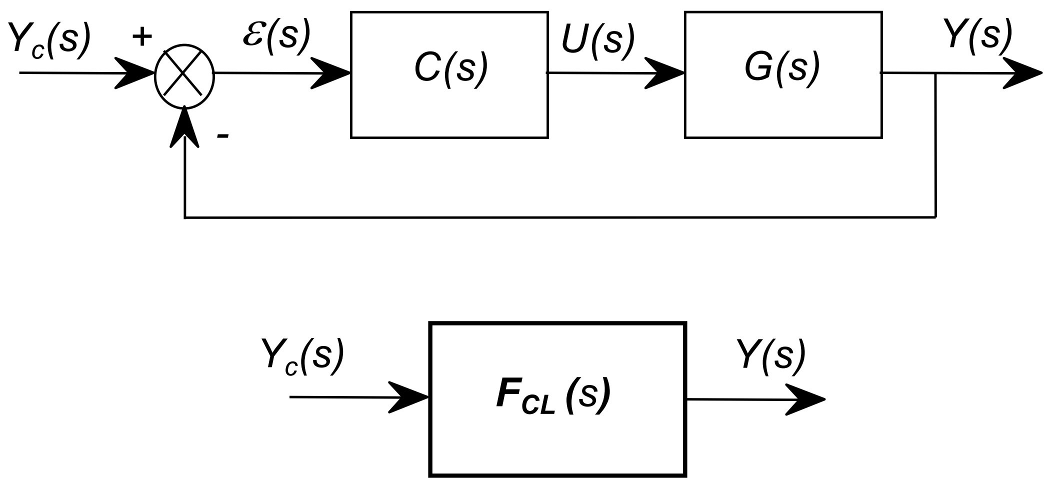
These slides have been modified from an initial version developed by Quanser

https://www.quanser.com

I sincerely thank Quanser for allowing me to adapt them

Closed-loop transfer function



We want $F_{CL}(s)$ to behave like a reference model $\boldsymbol{F_{ref}(s)}$

which takes, usually, the form of a standard second-order transfer function model

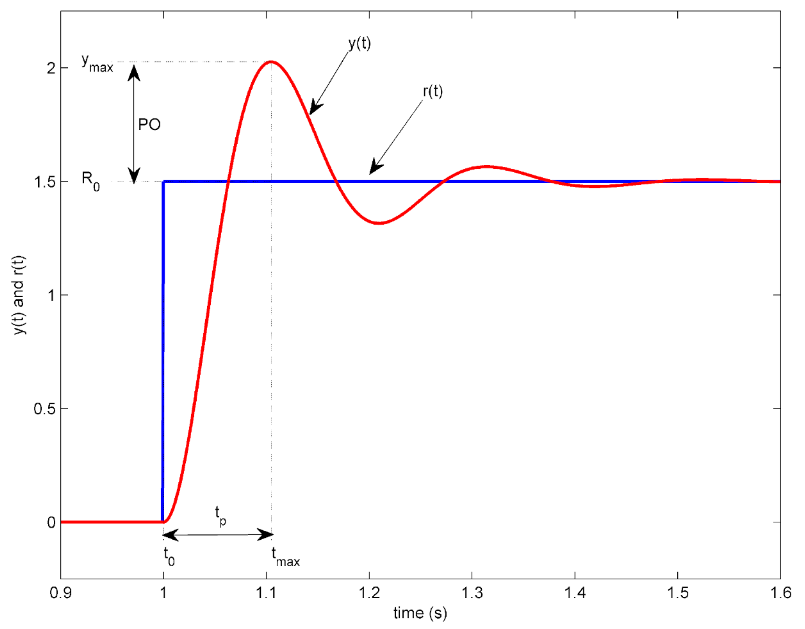
Standard 2nd order model

- Recall the standard second-order transfer function:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

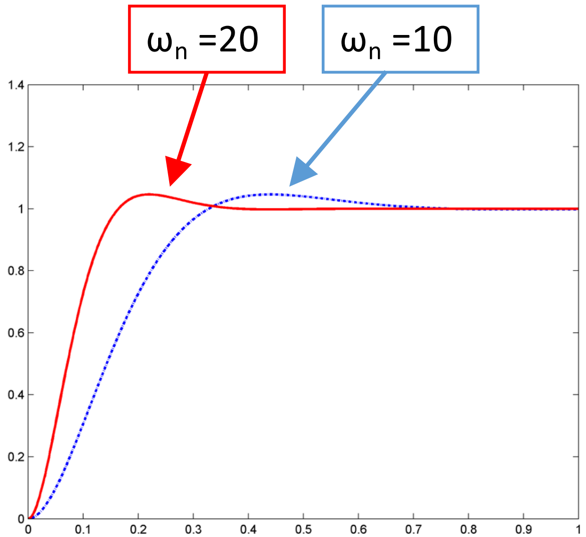
- ω_n is natural frequency
- ζ is the damping ratio

Step response of 2nd order system



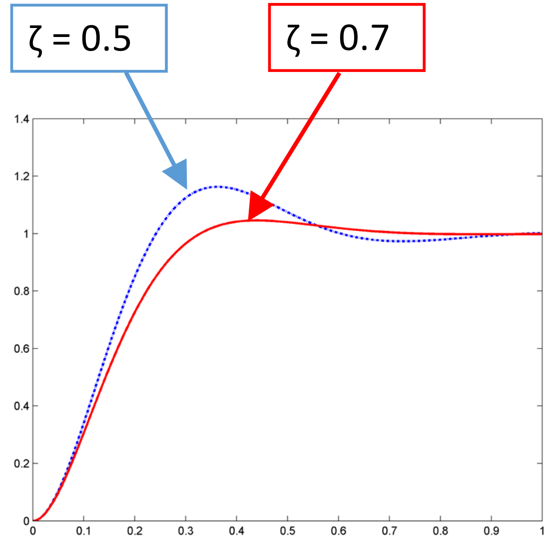
Effect on response?

Natural frequency effects the response speed



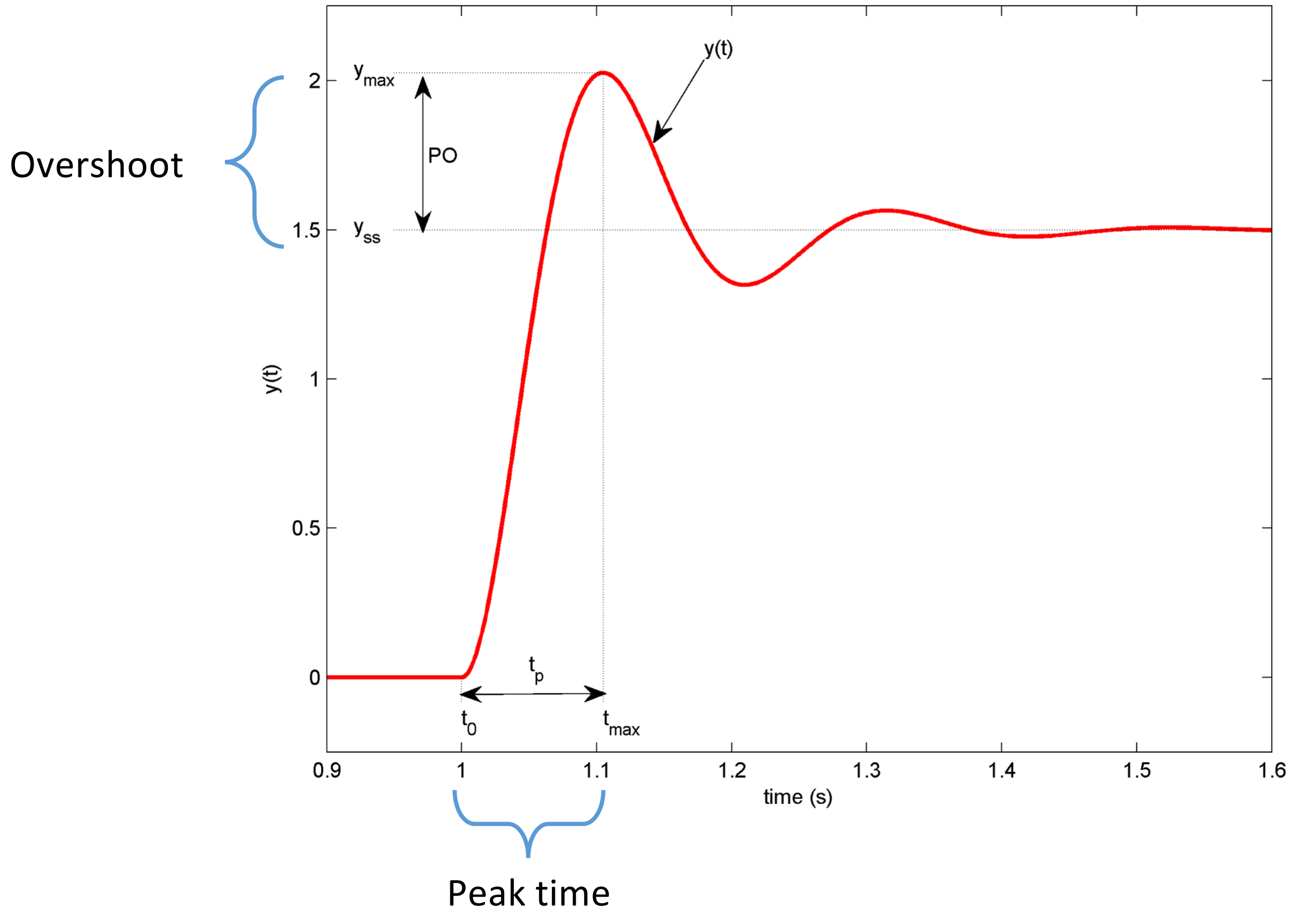
Increasing the natural frequency makes the response faster

Damping ratio effects the response shape



System is known as being **critically damped** when $\zeta = 1$; there is *no overshoot*

Peak Time and Overshoot



Finding Overshoot and Peak Time

Percent overshoot

- Measured from step response

$$PO = \frac{y_{max} - y_{ss}}{y_{ss}} \times 100$$

- Calculated from transfer function

$$PO = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

Peak time

- measured from step response

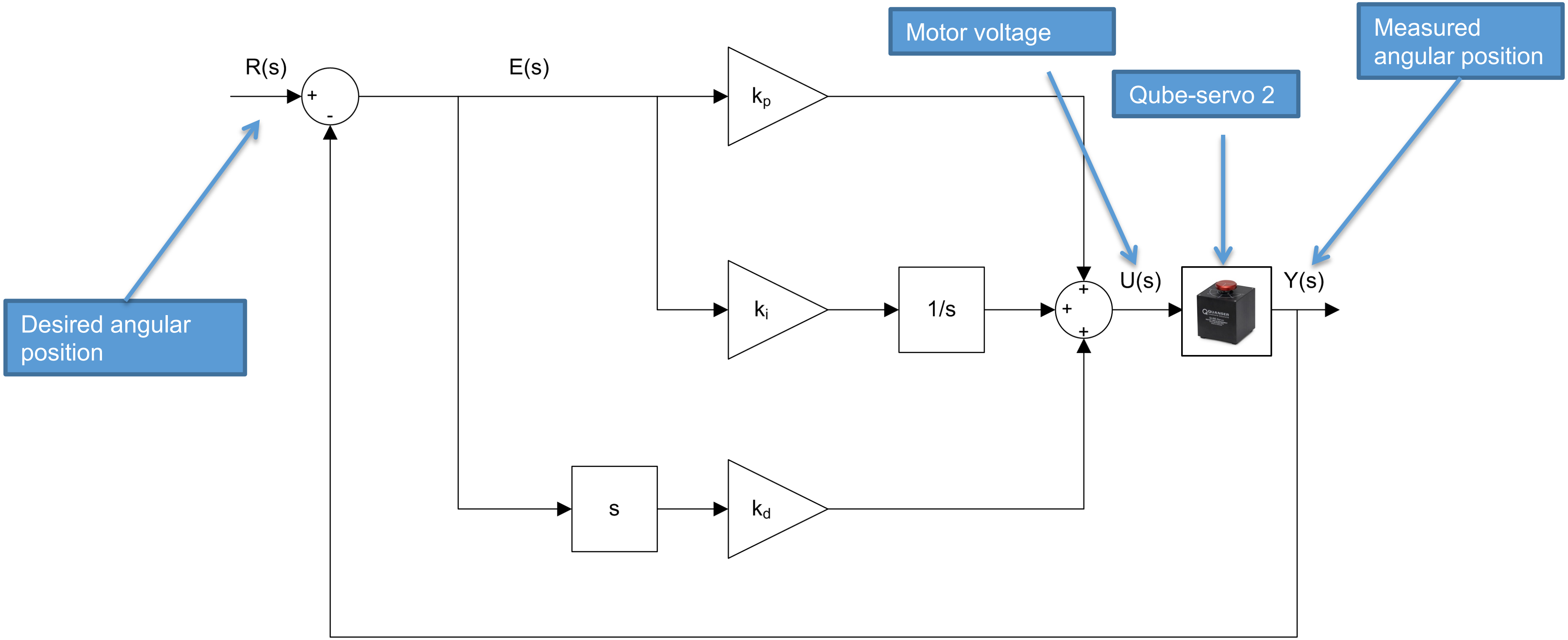
$$t_p = t_{max} - t_0$$

- Calculated from transfer function

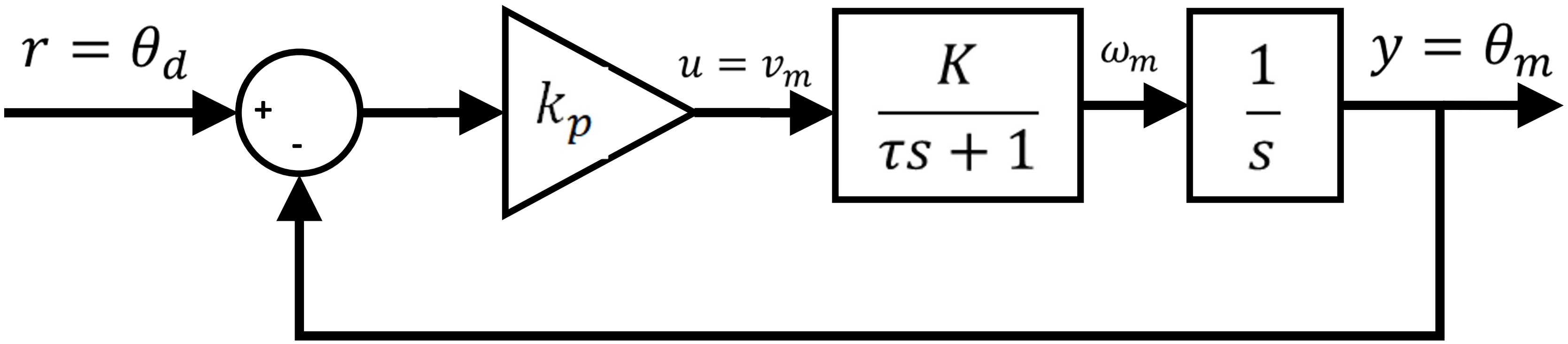
$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

PID control design

Example: Tuning for the QUBE-Servo 2 angular position



Simple Proportional Control



P Control Design

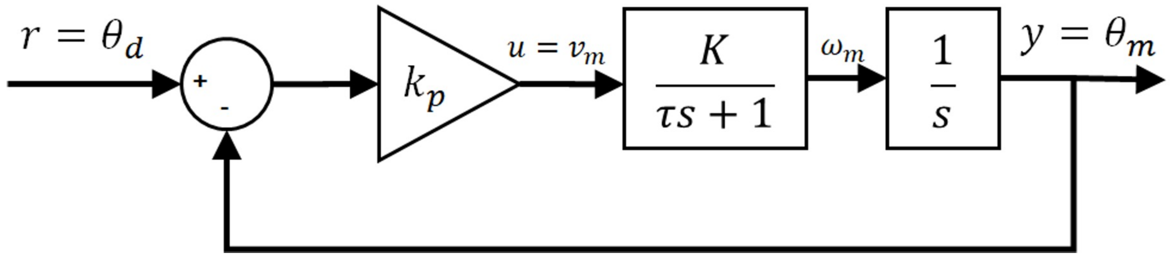
Closed-loop Response to Setpoint

H. Garnier

Closed-Loop Transfer Function

- Plant model

$$\frac{\Theta_m(s)}{V_{m(s)}} = \frac{K}{s(\tau s + 1)}$$



- Control input (i.e. voltage)

$$V_m(s) = k_p(\theta_d - \theta_m)$$

- Find closed-loop transfer function

Find Closed-Loop Transfer Function

- Closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{Kk_p/\tau}{s^2 + \frac{1}{\tau}s + \frac{Kk_p}{\tau}}$$

- Standard second-order transfer function

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = \frac{Kk_p}{\tau}$$

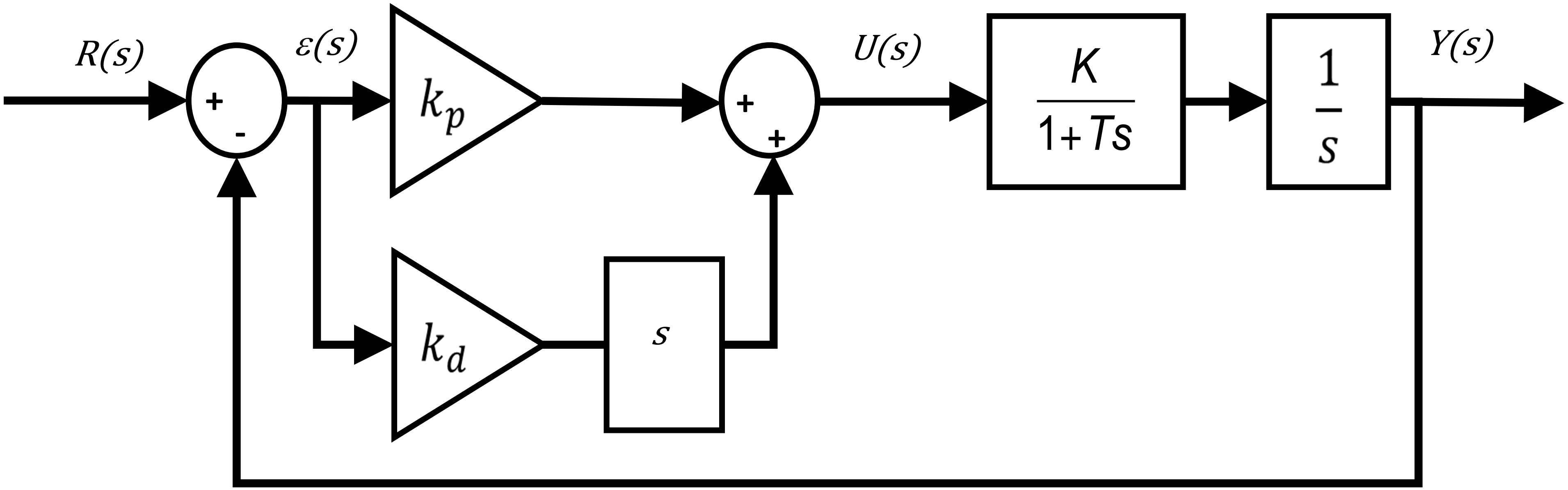
PD Control Design

Closed-loop Response to Setpoint

H. Garnier

Standard PD Control with P and D on the error

Implement this standard version of the PD control on the servo



Closed-Loop Transfer Function

- Given the plant model $$\frac{Y(s)}{U(s)} = \frac{K}{s(Ts+1)}$$

- and the compensator $$C(s) = k_p + k_d s$$

- Find closed-loop transfer function $$\frac{Y(s)}{R(s)}$$

Resulting Equation

- For PD control, the closed-loop equation of the QUBE is:

$$\frac{Y(s)}{R(s)} = \frac{K(k_p + k_d s)}{\tau s^2 + (Kk_d + 1)s + Kk_p}$$

- Standard second-order transfer function

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Do these equations match?

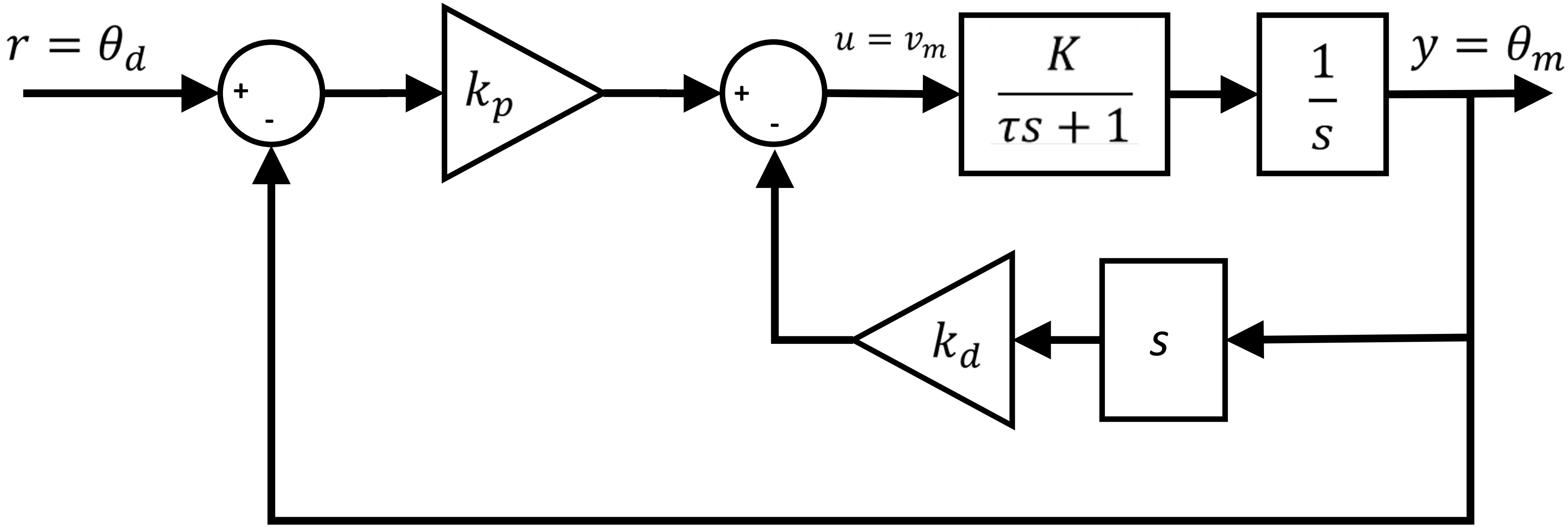
They do NOT match

- Closed-loop transfer function of the standard PD control does NOT match
- It has *zero* in numerator
- Gains designed for the standard PD controllers may not yield desired response on actual system

$$\frac{Y(s)}{R(s)} = \frac{K(k_p + k_d s)}{\tau s^2 + (Kk_d + 1)s + Kk_p}$$

PD Control with derivative on the output

Implement this variation of PD control on the servo, i.e. proportional control with derivative feedback



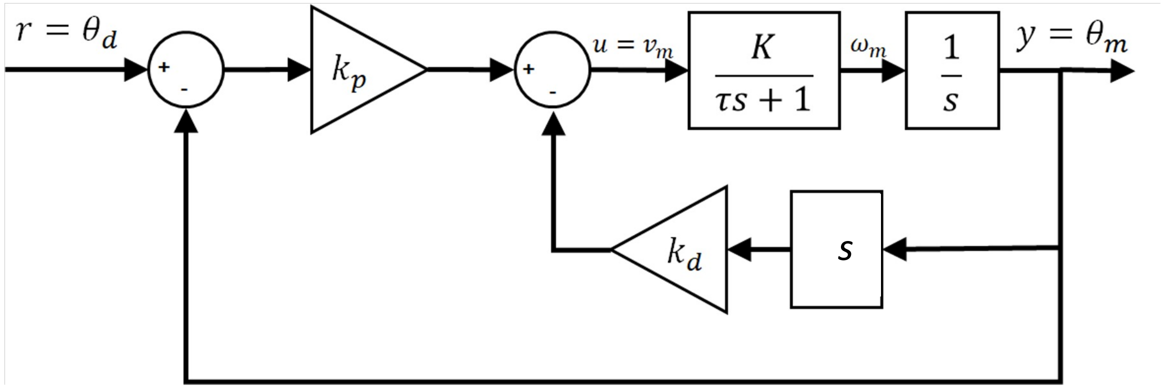
Closed-Loop Transfer Function

- Plant model

$$\frac{Y(s)}{U(s)} = \frac{K}{s(Ts+1)}$$

- Control input (i.e. voltage)

$$V_m(s) = k_p(\theta_d - \theta_m) - k_d s \theta_m$$



- Find closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{K k_p/\tau}{s^2 + \frac{(1+Kk_d)s}{\tau} + \frac{Kk_p}{\tau}}$$

Find Closed-Loop Transfer Function

Closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{Kk_p/\tau}{s^2 + \frac{(1+Kk_d)s}{\tau} + \frac{Kk_p}{\tau}}$$

Prototype second-order transfer function

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = \frac{Kk_p}{\tau}$$

$$2\zeta\omega_n = \frac{1+Kk_d}{\tau}$$

$$k_p = \frac{\omega_n^2\tau}{K}$$

$$k_d = \frac{2\zeta\omega_n\tau - 1}{K}$$

PD Design for (ω_n, ζ)

1. Based on required peak time and overshoot get ω_n and ζ

2. Given ω_n and ζ, what PD gains would I need?

$$k_p = \frac{\omega_n^2 \tau}{K}$$

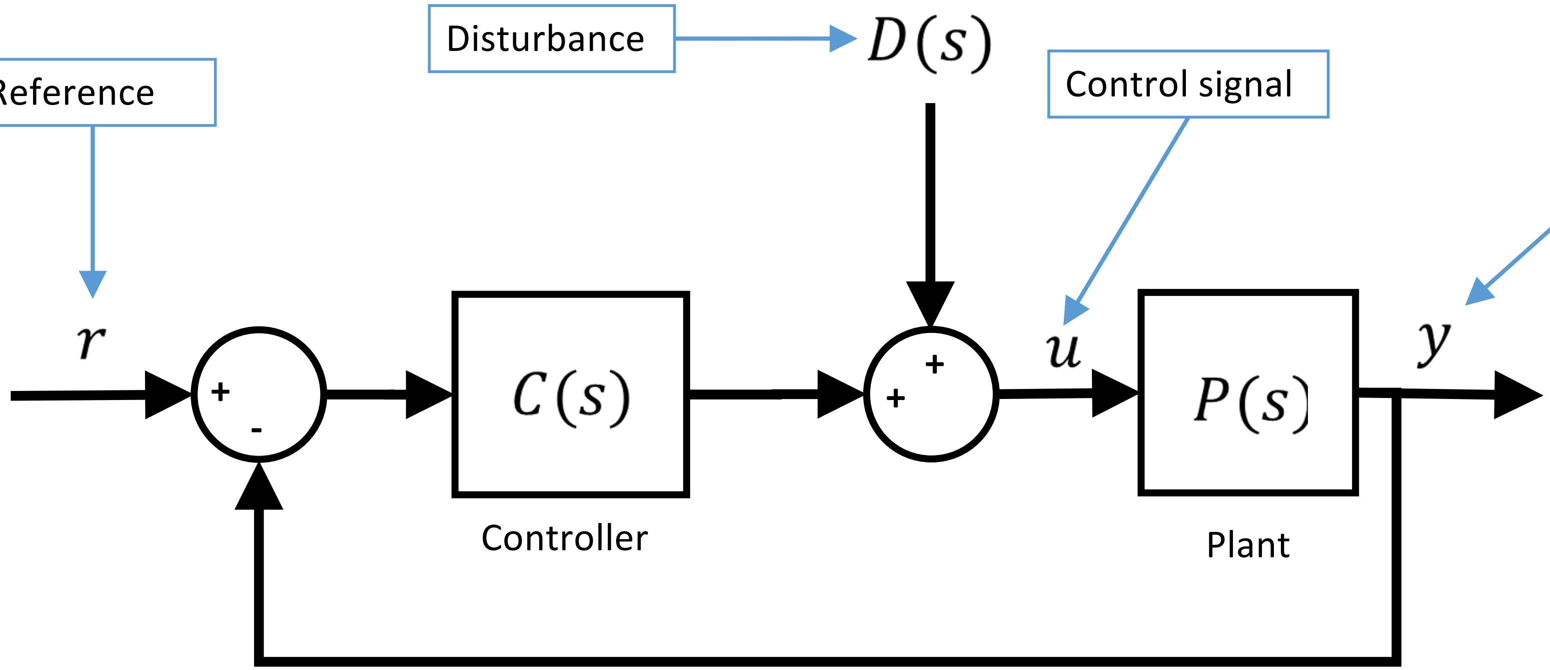
$$k_d = \frac{2\zeta\omega_n \tau - 1}{K}$$

PI Control Design

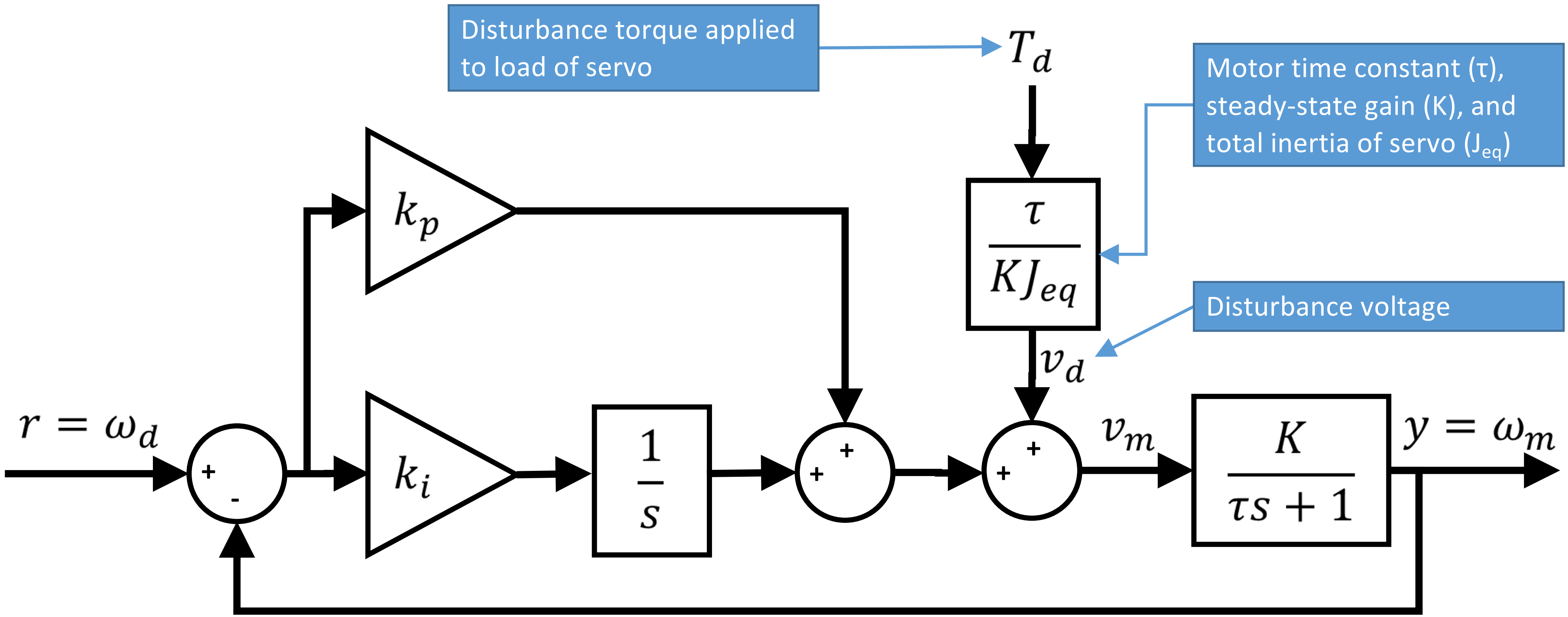
Closed-loop Response to Disturbances

H. Garnier

General Control System Block Diagram

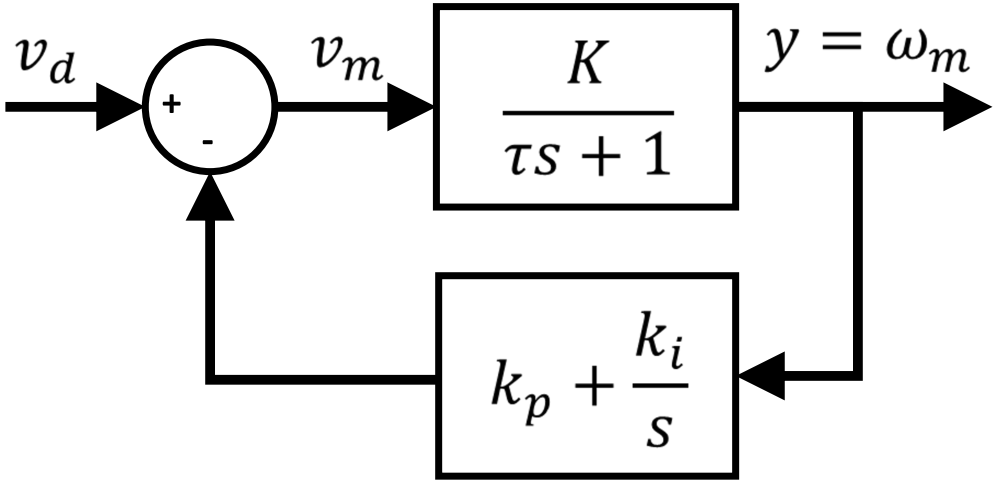


PI Control with Reference and Disturbance



Response to Load Disturbances

When referene command is zero, i.e. $r = 0$, block diagram can be reduced as follows



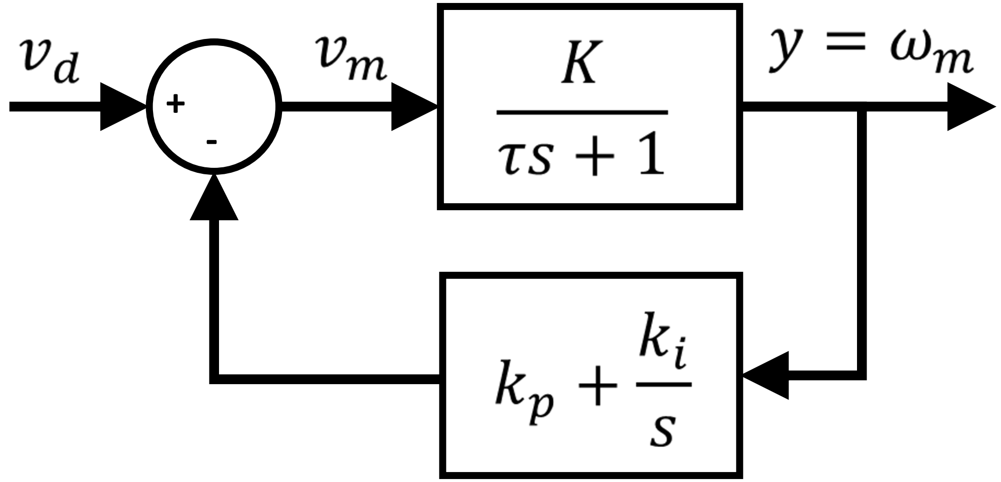
Find Closed-Loop Transfer Function

$$Y(s) = P(s)\big(V_d(s) - C(s)Y(s)\big)$$

$$Y(s) = \frac{P}{1+PC}V_d(s)$$

$$Y(s) = \frac{K}{\tau s + 1 + K\left(k_p + \frac{k_i}{s}\right)}V_d(s)$$

$$Y(s) = \frac{\frac{K}{\tau}s}{s^2 + \frac{1+Kk_p}{\tau}s + \frac{Kk_i}{\tau}}V_d(s)$$



Response to a Step Disturbance

$$V_d(s) = \frac{A_d}{s}$$

Apply

$$\Omega_m(s) = \frac{\frac{K}{\tau}s}{s^2 + \frac{1+Kk_p}{\tau}s + \frac{Kk_i}{\tau}} V_d(s)$$

Proportional control

$$Y(s) = \frac{A_d K/\tau}{s^2 + \frac{1+Kk_p}{\tau}s}$$

$$\omega_{ss} = \lim_{s\to 0} s\,Y(s) = \lim_{s\to 0} \frac{A_d K/\tau}{s + \frac{1+Kk_p}{\tau}}$$

$$\omega_{ss} = \frac{A_d K}{1+Kk_p}$$

PI control

$$Y(s) = \frac{A_d K/\tau}{s^2 + \frac{1+Kk_p}{\tau}s + \frac{Kk_i}{\tau}}$$

$$\omega_{ss} = \lim_{s\to 0} s\,Y(s) = \lim_{s\to 0} \frac{A_d K s/\tau}{s^2 + \frac{1+Kk_p}{\tau}s + \frac{Kk_i}{\tau}}$$

$$\omega_{ss} = 0$$

PID Control Design

Closed-loop Response to Setpoint and Disturbances

Automatique

H. Garnier

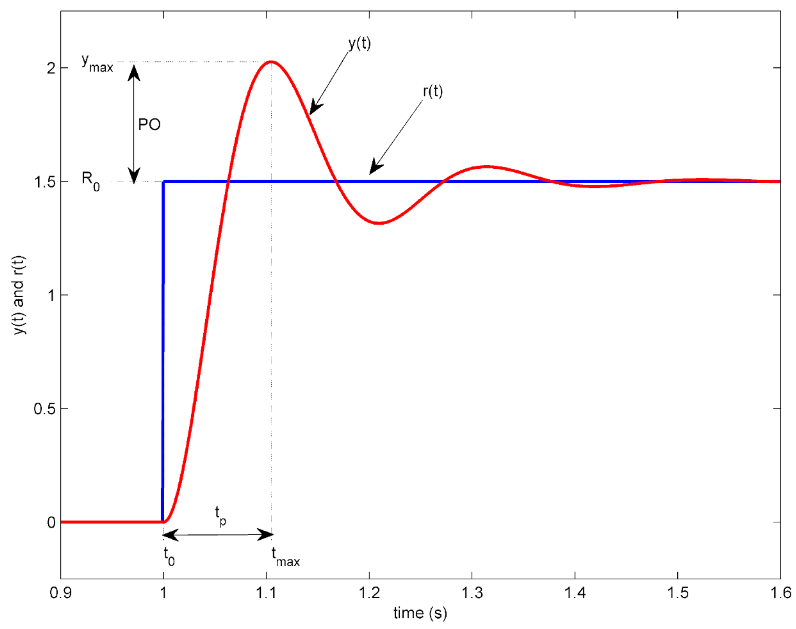
Standard 3rd Order System

- Characteristic 3rd order transfer function:

$$(s + p_0)(s^2 + 2\zeta\omega_n + \omega_n^2)$$

- ω_n is natural frequency
- ζ is the damping ratio
- p_0 is the location of a pole

Step response of 2nd order system



Closed-Loop Transfer Function

- Plant model

$$P(s) = \frac{\Theta_m(s)}{V_{m(s)}} = \frac{K}{s(\tau s + 1)}$$

- Control input (i.e. voltage)

$$U(s) = C(s)(R(s) - Y(s))$$

- Controller

$$C(s) = k_p + \frac{k_i}{s} + k_d s$$

- Find closed-loop transfer function

$$Y(s) = \frac{P(s)}{1 + P(s)C(s)} D(s) + \frac{P(s)C(s)}{1 + P(s)C(s)} R(s)$$

Find Closed-Loop Transfer Function

Closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{\frac{K}{\tau}(k_d s^2 + k_p s + k_i)}{s^3 + \frac{(1+Kk_d)}{\tau}s^2 + \frac{Kk_p}{\tau}s + \frac{Kk_i}{\tau}}$$

3rd order characteristic equation

$$(s+p_0)(s^2+2\zeta\omega_n+\omega_n^2)$$

$$= s^3 + (2\zeta\omega_n + p_0)s^2 + (\omega_n^2 + 2\zeta\omega_n p_0)s + \omega_n^2 p_0$$

$$\omega_n^2 p_0 = \frac{Kk_i}{\tau}$$

$$\omega_n^2 + 2\zeta\omega_n p_0 = \frac{Kk_p}{\tau}$$

$$2\zeta\omega_n + p_0 = \frac{1+Kk_d}{\tau}$$

PID Design for (ω_n, ζ, p_0)

1. Based on required peak time and overshoot get ω_n and ζ

2. Given p_0, ω_n and ζ, find PID gains needed

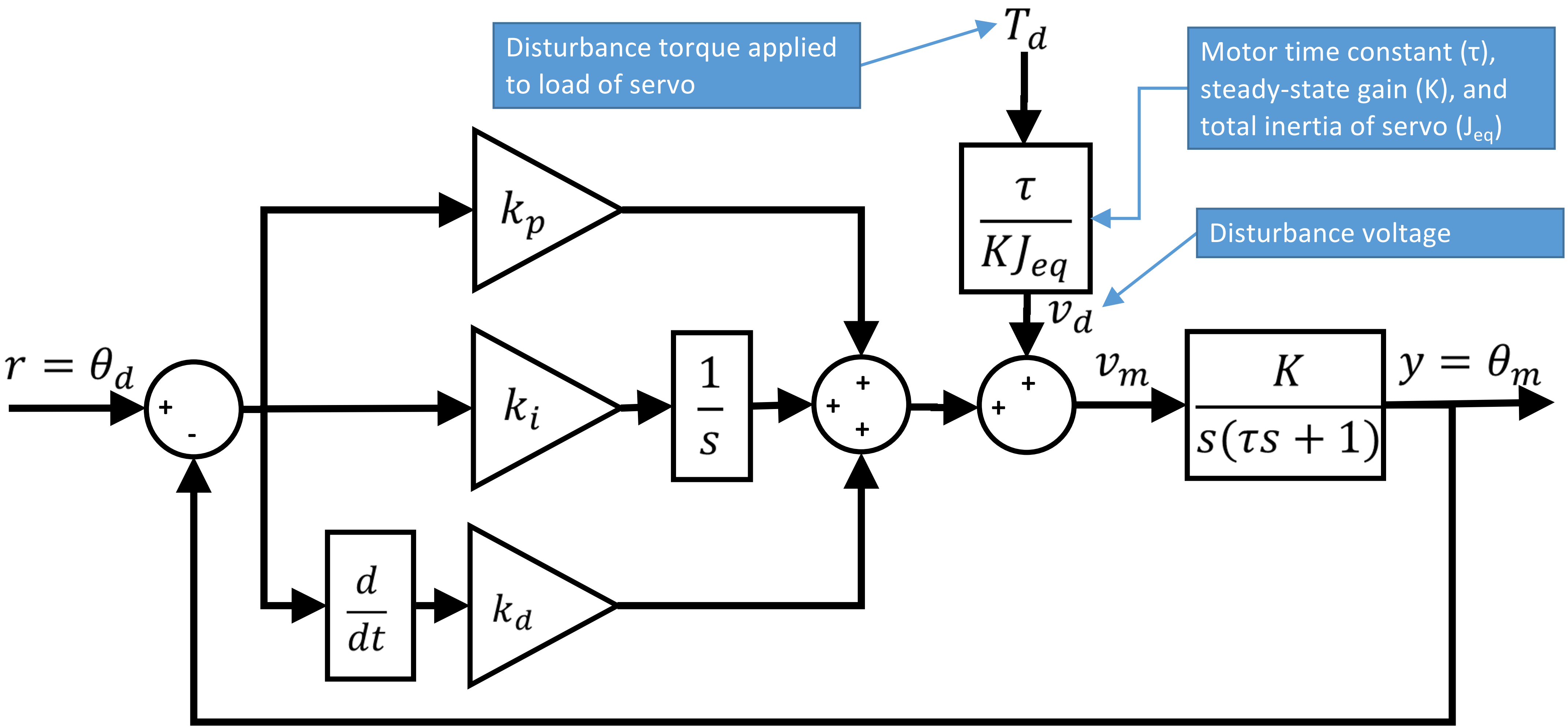
Note: the extra pole p_0 is a design that the user can adjust

$$k_p = \frac{\tau(\omega_n^2 + 2\zeta\omega_n p_0)}{K}$$

$$k_d = \frac{\tau(2\zeta\omega_n + p_0) - 1}{K}$$

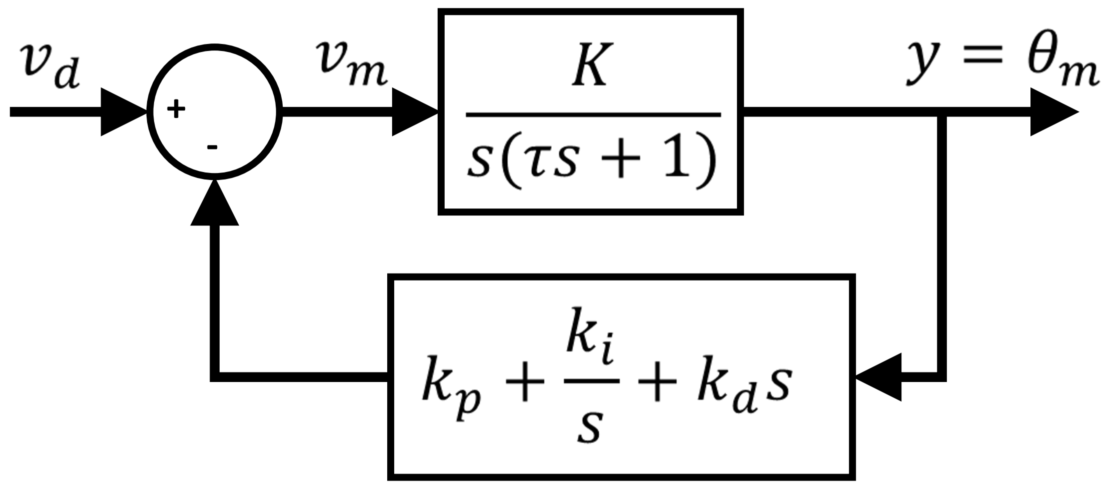
$$k_i = \frac{\omega_n^2 p_0 \tau}{K}$$

PID Control with Reference and Disturbance



Response to Disturbances

When reference command is zero, i.e. $r = 0$, block diagram can be reduced



Find Closed-Loop Transfer Function

$$Y(s) = P(s)\big(V_d(s) - C(s)Y(s)\big)$$

$$Y(s) = \frac{P}{1+PC}V_d(s)$$

$$Y(s) = \frac{\frac{K}{s(\tau s+1)}}{1+\frac{K}{s(\tau s+1)}\left(k_p+\frac{k_i}{s}+k_d s\right)}V_d(s)$$

$$Y(s) = \frac{Ks}{s(\tau s+1)+K\left(k_p s+k_i+k_d s^2\right)}V_d(s)$$

$$Y(s) = \frac{\frac{K}{\tau}s}{s^3+\frac{(1+Kk_d)}{\tau}s^2+\frac{Kk_p}{\tau}s+\frac{Kk_i}{\tau}}V_d(s)$$

Response to a Step Disturbance

$$V_d(s) = \frac{A_d}{s}$$

Apply

$$\Theta_m(s) = \frac{\frac{K}{\tau}s}{s^3 + \frac{(1+Kk_d)}{\tau}s^2 + \frac{Kk_p}{\tau}s + \frac{Kk_i}{\tau}}V_d(s)$$

PD control

$$Y(s) = \frac{\frac{K}{\tau}}{s^2 + \frac{1+Kk_d}{\tau}s + \frac{Kk_p}{\tau}}V_d(s)$$

$$\theta_{ss} = \lim_{s\to 0} s\,Y(s) = \lim_{s\to 0} s\frac{\frac{A_d}{s}\frac{K}{\tau}}{s + \frac{1+Kk_p}{\tau}}$$

$$\theta_{ss} = \frac{A_d K}{1+Kk_p}$$

PID control

$$\Theta_m(s) = \frac{\frac{K}{\tau}s}{s^3 + \frac{(1+Kk_d)}{\tau}s^2 + \frac{Kk_p}{\tau}s + \frac{Kk_i}{\tau}}V_d(s)$$

$$\theta_{ss} = \lim_{s\to 0} s\,Y(s) = \lim_{s\to 0} s\frac{A_d/s \cdot Ks/\tau}{s^3 + \frac{(1+Kk_d)}{\tau}s^2 + \frac{Kk_p}{\tau}s + \frac{Kk_i}{\tau}}$$

$$\theta_{ss} = 0$$